A Study Of Fault-Tolerant Quantum Computing with Qudits

Synopsis

Submitted in the partial fulfillment for the requirements of

Doctor of Philosophy

in

Physics

Submitted by

Kanumuri Lakshmi Ramya

Under the supervision of

Dr. Shiroman Prakash

Department of Physics and Computer Science

Department of Physics and Computer Science
Faculty of Science
Dayalbagh Educational Institute (Deemed University)
Dayalbagh, Agra-282005

2022

1 Introduction

Quantum computing

Quantum computing can solve complex and classically unattainable problems as it uses properties like superposition and entanglement for performing computation. An early prediction about quantum computing is given by Paul Benioff [1], where he gave a quantum mechanical model of the turning machine. Later, Richard Feynman and Yuri Manin theorized the potential of a quantum computer to simulate classically intractable problems [2, 3]. Classical complexity theory classifies problems as classically tractable and intractable based on the criteria, if an algorithm exists to solve them with the resources (time and memory), which are polynomial in the size of the input. Famous classically intractable problems are factoring n-bit number and travelling salesman problem (finding the minimum cyclic path connecting n cities with specified distances). The above two examples fall into the non-deterministic polynomial of the complexity class.

Travelling salesman problem is an NP-complete problem. Any NP problem can be transformed into an instance of an NP-complete problem in polynomial time. An NP problem can not be solved in polynomial time, but a given solution can be verified in polynomial time. Some of the issues which can be solved explicitly by a quantum computer are RSA public-key cryptography (which depends on the difficulty of factoring large numbers [4]) and simulating quantum systems is another classically intractable problem. To represent a state in a quantum computer with n qubits requires a complex vector with 2^ncomponents. While a classical n-bit digital computer also has 2^n possible states, but requires only an n-dimensional state space, as an n component binary vector can describe a state.

Unlike classical systems, quantum systems are hard to simulate, requiring exponential classical resources. They generically utilize the entire 2^n-dimensional Hilbert space as they evolve. To realize the full potential of quantum computing, we need to address issues like decoherence and noise from the environment. The above facts inspired Feynman to infer that quantum computers that used quantum mechanics intrinsically might be more powerful than their classical counterparts [2]. It provides a possibility that classical complexity classes might not be relevant for quantum computers. Shor's algorithm [5] for factoring numbers on a quantum computer in polynomial time is evidence of

the same. Similarly, Grover's algorithm [6] is used to find a single object in an unsorted database of N objects in $O(\sqrt{N})$ time on a quantum computer. While the same would require O(N) time on a classical computer. Quantum computers are more susceptible to errors compared to modern classical computers. Classical computers guard against errors by being digital instead of analogue.

2 Literature Review

2.1 Quantum error correction

Quantum error correction protects quantum information from decoherence and quantum noise. A stark difference between classical and quantum computing is that classical information (bits) can be copied to prevent errors and perform fault tolerance. In contrast, quantum computing can not be copied due to the 'No-cloning theorem' [7]. Qubits, i.e., entangled states, are incredibly fragile, and any measurement would collapse to less entangled states. Small interactions from the environment imply continuous measurement of the system and will grow with the size of the system, making it impossible to ignore. The system will decohere and appear as a classical system. Isolation of the quantum states from the external environment can only be attained to a specific limit but can not be completely eliminated. Quantum error-correcting codes were first formulated by Shor [8] and Steane [9] to address these issues. Unlike the classical computer, where we need to save only the bit values, quantum computers also need to preserve the phase information in entangled states. Therefore quantum error-correcting codes require a new approach.

2.2 Fault-tolerant quantum computing

Noise from the environment and decoherence is not only the sources of errors in a quantum computer. The logic gates, which are eventually built from qubits to perform logical operations, also introduce errors. The code stabilizer [10-12] is known for its general formalism for quantum error-correcting codes. Fault-tolerance operations aim to prevent the spread of errors within a block. Quantum gates spread the errors in both forward and backward directions, making them different from a classical gate. Daniel Gottesman was among the first scientists to develop a theory of fault-tolerant quantum computing [13], where he demonstrated fault-tolerant universal computation for any stabilizer code. Using qudits of prime dimensions $p \neq 2$ [14] is a new approach to fault-tolerant quantum computing for both theoretical and practical reasons. Contextuality, which is realized as a

necessary feature of quantum mechanics to execute the full potential of quantum computer [15], was possible by considering the magic state distillation of qudits of odd prime dimension. High thresholds for magic state distillation were experimentally observed for magic state distillation schemes in odd prime dimension [16-18]. Qutrits (and qudits) encoded in the orbital angular momentum states of photons [19], implementation of a single-qutrit quantum computer via trapped ions [20], and demonstration of control and tomography of a three-state artificial atom [21] are few experimental realizations of quantum computing using qudits with a dimension greater than 2.

2.3 Magic state distillation

Magic state distillation is an approach to achieve fault-tolerant quantum computing [16] which relies on ancilla qudits in non-stabilizer states known as magic states to promote the Clifford group to universal quantum computation. Magic state distillation for qudits is less explored in comparison to qubits. Practical advantages of qudits for magic stated distillation was demonstrated in [18]. Qudits of odd prime dimensions have been studied for the past few years [16, 17, 22]. Theoretically, qudit magic state distillation has been used to identify contextuality as a necessary, and possibly sufficient resource for universal quantum computation [15, 23]. For the experimental realization of the quantum computer, one must invoke the methods of fault-tolerant quantum computing [13].

3 Methodology

3.1 Qudits of odd prime dimensions

Despite the pertaining challenges, there is progress towards fault-tolerant quantum computing [23-31]. Surface code [32-35], where Clifford unitaries can be implemented in a vigorous fault-tolerant way and having very high error thresholds, is considered a prime framework for fault tolerance. The critical aspect of being addressed is the Eastin-Knill theorem [36], as the theorem states that a universal set of transversal gates is not possible. We know that Clifford unitaries can be realized transversally[37, 38], but this does not address the Eastin-Knill theorem in totality. Quantum states which are called the magic states, when injected, can promote the Clifford group to universal quantum computing[39]. Nevertheless, we need to address that magic states are invariably noisy. Therefore, there is a need for protocols, such as stabilizer protocols used to purify many copies of the magic states. It will help to improve the overall performance of the induced quantum gates[39-41]. A salient

factor to consider is the overhead on purifying many magic states into less noisy ones. Bravyi-Haan qubit protocol is one such protocol where quadratic reduction of noise per cycle is demonstrated. There is progress in improving the distillation rates[40, 42-44] while developing protocols for qudits of odd prime dimensions[16, 17, 22, 45, 46]. Magic protocols, when analyzed from the perspective of magic monotones, provide an upper bound on the distillation rates. It is also used to analyze gate synthesis[47-51].

3.2 Phase Space Representation of Quantum states

One of the main aim is the representation of any quantum system or operation on a d-dimensional system on a discrete phase space in terms of quasi probabilities[52]. This construction is the discrete version of wigner representation in quantum optics[53, 54].Consider a d- dimensional quantum system with Hilbert space H_d and let $0,1,\dots d-1$ represent the standard computational basis over $Z_d = 0,1,\dots d-1$. Generalized pauli matrices X, Z in this space i.e the shift and phase operators respectively, act on the basis states as,

$$X(K) = K + 1$$

$$Z(K) = w^k k$$

where $w := e^{2\pi i/d}$ is the d-th root of unity and addition is taken modulo d. A phase space $P := Z_d \times Z_d$ can be constructed which gives a complete representation of the quantum system. Displacement operator at a point $Z := (q, p)$ can be defined as

$$D_z := \tau^{qp} X^q Z^p$$

the phase space operator $\tau := -\omega^{1/2}$ is for the purpose of unitarity. Displacement operators for a composite system with dimensions $d_{tot} = d_1 \dots d_n$ and Hilbert space $H_{d_1} \otimes \dots \otimes H_{d_n}$ can be defined as

$$D_z := D_{(q_1,p_1)} \otimes \dots \otimes D_{(q_n,p_n)}$$

Phase space point for a composite system can be represented as

$$Z := (q_1 \dots . q_n, p_1 \dots p_n) \in P_{d_1} \times \dots \times P_{d_n}$$

3.3 Wigner Representation for the Quantum States and Quantum Operations

Hermisian observables which change covarients under clifford operations can be defined to

get a total decomposition of arbitrary quantum states and quantum operations.Phase point operator at every point $Z \in P_d$ can be defined as

$$A_z := \frac{1}{d}\sum_{y\in p_d} w^{\eta(z,y)} D_y$$

and $\eta(z,y)$ is the symplectic inner product between any two points $z, y \in P_d$ is given by

$$\eta(z,y) := y^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} z,$$

where 0,1 denote the $n \times n$ zero and identity matrices. Any quantum state ρ can be expressed as a linear combination of the phase point operators as they form orthogonal operator basis with respect to the Hilbert-Schmidt inner product. $\rho = \sum_{z\in p_d} W_\rho(z) A_z$, and the coefficients $W_\rho(z)$is the Wigner distribution of state ρ,

$$W_\rho(z) := \frac{1}{d} tr[A_z \rho].$$

3.4 Magic theories for quantum computing

Numerous magic theories are present, where magic states are computational resource states with respect to a class of quantum operations that are considered free[55]. Among the other candidates, Clifford unitaries, Pauli measurements, and the ability to discard quantum systems fall under the natural class of free operations[56-58]. Magic monotones are used for bounding distillation rates in any magic theory. A magic monotone is a monotonically non-increasing real-valued function of a quantum state M(ρ) under the free operations of the magic theory. Mana of a state is a marked monotone[59], which is defined as

$$M(\rho) := log(2Sn(\rho) + 1);$$

Here the sum negativity [59] is the sum of the negative components in W_ρ,

$$sn(\rho) := \sum_{z:W_\rho(z)<0} |W_\rho(z)|.$$

Given the fact that $W_\rho(z)$ is a normalized quasi distribution, it can be written as $M(\rho) = \sum_z |W_\rho(z)|$.

Mana is an additive monotone that is non-increasing under free operations, and this fact gives a constraint on magic state interconvertibility.

4 Objectives

The following objectives will be pursued.

1) Further develop and apply the resource theory of magic for qudits of odd prime dimensions.
2) Investigate implications of resource theory in magic state distillation for qudits of odd prime dimensions.
3) Search for new magic state distillation routines, with focus on qudits of odd prime dimensions.

References

[1] P. Benioff, "The computer as a physical system: A microscopic quantum mechanical hamiltonian model of computers as represented by turing machines," Journal of statistical physics, vol. 22, no. 5, pp. 563–591, 1980.

[2] R. P. Feynman, "Simulating physics with computers," Int. J. Theor. Phys, vol. 21, no. 6/7, 1982.

[3] Y. I. Manin, "Vychislimoe i nevychislimoe (computable and noncomputable), moscow: Sov,"1980.

[4] R. L. Rivest, A. Shamir, and L. Adleman, "A method for obtaining digital signatures and public-key cryptosys tems," Communications of the ACM, vol. 21, no. 2, pp. 120–126, 1978.

[5] P. W. Shor, "Algorithms for quantum computation: discrete logarithms and factoring," in Proceedings 35th annual symposium on foundations of computer science. Ieee, 1994, pp. 124–134.

[6] L. K. Grover, "A fast quantum mechanical algorithm for database search," in Proceedings of the twenty-eighth annual ACM symposium on Theory of computing, 1996, pp. 212–219.

[7] W. K. Wootters and W. H. Zurek, "A single quantum cannot be cloned," Nature, vol. 299, no. 5886, pp. 802–803, 1982.

[8] D. P. DiVincenzo and P. W. Shor, "Fault-tolerant error correction with efficient quantum codes," Physical review letters, vol. 77, no. 15, p. 3260, 1996.

[9] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, "Mixed-state entanglement and quantum error correction," Physical Review A, vol. 54, no. 5, p. 3824, 1996.

[10] D. Gottesman, "Class of quantum error-correcting codes saturating the quantum hamming bound," Physical Review A, vol. 54, no. 3, p. 1862, 1996.

[11] A. R. Calderbank, E. M. Rains, P. W. Shor, and N. J. Sloane, "Quantum error correction and orthogonal geometry," Physical Review Letters, vol. 78, no. 3, p. 405, 1997.

[12] A. R. Calderbank, E. M. Rains, P. Shor, and N. J. Sloane, “Quantum error correction via codes over gf (4),”IEEE Transactions on Information Theory, vol. 44, no. 4, pp. 1369–1387, 1998.

[13] D. Gottesman, “Theory of fault-tolerant quantum computation,” Physical Review A, vol. 57, no. 1, p. 127, 1998.

[14] “Fault-tolerant quantum computation with higher-dimensional systems,” in NASA International Conference on Quantum Computing and Quantum Communications. Springer, 1998, pp. 302–313.

[15] M. Howard, J. Wallman, V. Veitch, and J. Emerson, “Contextuality supplies the ‘magic’for quantum computation,” Nature, vol. 510, no. 7505, pp. 351–355, 2014.

[16] H. Anwar, E. T. Campbell, and D. E. Browne, “Qutrit magic state distillation,” New Journal of Physics, vol. 14,no. 6, p. 063006, 2012.

[17] E. T. Campbell, H. Anwar, and D. E. Browne, “Magic-state distillation in all prime dimensions using quantum reed-muller codes,” Physical Review X, vol. 2, no. 4, p. 041021, 2012.

[18] E. T. Campbell, “Enhanced fault-tolerant quantum computing in d-level systems,” Physical review letters, vol.113, no. 23, p. 230501, 2014.

[19] A. Mair, A. Vaziri, G. Weihs, and A. Zeilinger, “Entanglement of the orbital angular momentum states of photons,” Nature, vol. 412, no. 6844, pp. 313–316, 2001.

[20] A. Klimov, R. Guzman, J. Retamal, and C. Saavedra, “Qutrit quantum computer with trapped ions,” Physical Review A, vol. 67, no. 6, p. 062313, 2003.

[21] R. Bianchetti, S. Filipp, M. Baur, J. M. Fink, C. Lang, L. Steffen, M. Boissonneault, A. Blais, and A. Wallraff, “Control and tomography of a three level superconducting artificial atom,” Physical review letters, vol. 105, no. 22, p. 223601, 2010.

[22] H. Dawkins and M. Howard, “Qutrit magic state distillation tight in some directions,” Physical review letters, vol. 115, no. 3, p. 030501, 2015.

[23] E. T. Campbell, B. M. Terhal, and C. Vuillot, “Roads towards fault-tolerant universal quantum computation,” Nature, vol. 549, no. 7671, pp. 172–179, 2017.

[24] R. Raussendorf and H. J. Briegel, “A one-way quantum computer,” Physical Review Letters, vol. 86, no. 22, p.5188, 2001.

[25] R. Raussendorf, D. E. Browne, and H. J. Briegel, “Measurement-based quantum computation on cluster states,” Phys. Rev. A, vol. 68, p. 022312, Aug 2003.

[26] N. H. Nickerson, J. F. Fitzsimons, and S. C. Benjamin, “Freely scalable quantum technologies using cells of 5-to-50 qubits with very lossy and noisy photonic links,” Phys. Rev. X, vol. 4, p. 041041, Dec 2014.

[27] E. Nikahd, M. Sedighi, and M. Saheb Zamani, “Nonuniform code concatenation for universal fault-tolerant quantum computing,” Phys. Rev. A, vol. 96, p. 032337, Sep 2017.

[28] R. Chao and B. W. Reichardt, “Fault-tolerant quantum computation with few qubits,” npj Quantum Information, vol. 4, no. 1, pp. 1–8, 2018.

[29] C. Lin, G. Yang, Q. Luo, and X. Li, “Pieceable fault tolerant conversion between 5-qubit code and 7-css code,” Quantum Information Processing, vol. 19, no. 8, pp. 1–21, 2020.

[30] C. Lin and G. Yang, “Concatenated pieceable fault-tolerant scheme for universal quantum computation,” Physical Review A, vol. 102, no. 5, p. 052415, 2020.

[31] J. E. Bourassa, R. N. Alexander, M. Vasmer, A. Patil, I. Tzitrin, T. Matsuura, D. Su, B. Q. Baragiola, S. Guha, G. Dauphinais et al., “Blueprint for a scalable photonic fault-tolerant quantum computer,” Quantum, vol. 5, p.392, 2021.

[32] S. B. Bravyi and A. Y. Kitaev, “Quantum codes on a lattice with boundary,” arXiv preprint quant-ph/9811052, 1998.

[33] M. H. Freedman, “Quantum computation and the localization of modular functors,” Foundations of Computational Mathematics, vol. 1, no. 2, pp. 183–204, 2001.

[34] E. Dennis, A. Kitaev, A. Landahl, and J. Preskill, “Topological quantum memory,” Journal of Mathematical Physics, vol. 43, no. 9, pp. 4452–4505, 2002

[35] R. Raussendorf and J. Harrington, “Fault-tolerant quantum computation with high threshold in two dimensions,” Physical review letters, vol. 98, no. 19, p. 190504, 2007.

[36] B. Eastin and E. Knill, “Restrictions on transversal encoded quantum gate sets,” Physical review letters, vol.102, no. 11, p. 110502, 2009.

[37] A. R. Calderbank and P. W. Shor, “Good quantum error-correcting codes exist,” Physical Review A, vol. 54, no. 2, p. 1098, 1996.

[38] A. Steane, “Multiple-particle interference and quantum error correction,” Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences, vol. 452, no. 1954, pp. 2551–2577, 1996.

[39] S. Bravyi and A. Kitaev, “Universal quantum computation with ideal clifford gates and noisy ancillas,” Physical Review A, vol. 71, no. 2, p. 022316, 2005.

[40] C. Jones, “Multilevel distillation of magic states for quantum computing,” Physical Review A, vol. 87, no. 4, p. 042305, 2013.

[41] J. O’Gorman and E. T. Campbell, “Quantum computation with realistic magic-state factories,” Physical Review A, vol. 95, no. 3, p. 032338, 2017.

[42] J. Haah, M. B. Hastings, D. Poulin, and D. Wecker, “Magic state distillation with low space overhead and optimal asymptotic input count,” Quantum, vol. 1, p. 31, 2017.

[43] M. B. Hastings and J. Haah, “Distillation with sublogarithmic overhead,” Physical review letters, vol. 120, no. 5, p. 050504, 2018.

[44] D. Litinski, “Magic state distillation: Not as costly as you think,” Quantum, vol. 3, p. 205, 2019.

[45] A. Krishna and J.-P. Tillich, “Towards low overhead magic state distillation,” Physical review letters, vol. 123, no. 7, p. 070507, 2019.

[46] S. Prakash, “Magic state distillation with the ternary golay code,” Proceedings of the Royal Society A, vol. 476, no. 2241, p. 20200187, 2020.

[47] E. T. Campbell and M. Howard, “Unifying gate synthesis and magic state distillation,” Physical review letters, vol. 118, no. 6, p. 060501, 2017.

[48] “Unified framework for magic state distillation and multiqubit gate synthesis with reduced resource cost,”Physical Review A, vol. 95, no. 2, p. 022316, 2017.

[49] S. Prakash, A. Jain, B. Kapur, and S. Seth, “Normal form for single-qutrit clifford+ t operators and synthesis ofsingle-qutrit gates,” Physical Review A, vol. 98, no. 3, p. 032304, 2018.

[50] J. R. Seddon, B. Regula, H. Pashayan, Y. Ouyang, and E. T. Campbell, “Quantifying quantum speedups:Improved classical simulation from tighter magic monotones,” PRX Quantum, vol. 2, no. 1, p. 010345, 2021.

[51] L. Leone, S. F. Oliviero, and A. Hamma, “R\’enyi entropy of magic,” arXiv preprint arXiv:2106.12587, 2021.

[52] C. Ferrie and J. Emerson, “Frame representations of quantum mechanics and the necessity of negativity in quasi-probability representations,” Journal of Physics A: Mathematical and Theoretical, vol. 41, no. 35, p. 352001,2008.

[53] E. P. Wigner, “On the quantum correction for thermodynamic equilibrium,” in Part I: Physical Chemistry. PartII: Solid State Physics. Springer, 1997, pp. 110–120.

[54] A. Vourdas, “Quantum systems with finite hilbert space,” Reports on Progress in Physics, vol. 67, no. 3, p. 267,2004.

[55] E. Chitambar and G. Gour, “Quantum resource theories,” Reviews of Modern Physics, vol. 91, no. 2, p. 025001,2019.

[56] M. Ahmadi, H. B. Dang, G. Gour, and B. C. Sanders, “Quantification and manipulation of magic states,”Physical Review A, vol. 97, no. 6, p. 062332, 2018.

[57] J. R. Seddon and E. T. Campbell, “Quantifying magic for multi-qubit operations,” Proceedings of the Royal Society A, vol. 475, no. 2227, p. 20190251, 2019.

[58] X. Wang, M. M. Wilde, and Y. Su, “Quantifying the magic of quantum channels,” New Journal of Physics, vol. 21, no. 10, p. 103002, 2019.

[59] V. Veitch, S. H. Mousavian, D. Gottesman, and J. Emerson, “The resource theory of stabilizer quantum computation,” New Journal of Physics, vol. 16, no. 1, p. 013009, 2014.